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Black Holes in Bose–Einstein Condensates

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It is shown that there exist both dynamically stable and unstable dilute-gas Bose–Einstein condensates that, in the hydrodynamic limit, exhibit a behavior completely analogous to that of gravitational black holes. The dynamical instabilities involve creation of quasiparticle pairs in positive and negative energy states. We illustrate these features in two qualitatively different one-dimensional models. We have also simulated the creation of a stable sonic black hole by solving the Gross–Pitaevskii equation numerically for a condensate subject to a trapping potential that is adiabatically deformed. A sonic black hole could in this way be created experimentally with state-of-the-art or planned technology.

KEY WORDS: Bose–Einstein condensation; gravity analogs; black holes.

1. INTRODUCTION

The past decade has witnessed an increasing interest in simulating gravitational configurations and processes in condensed matter systems in the laboratory. The key observation was originally made by Unruh (1981, 1995) and further analyzed by Visser (1993, 1998a,b; Liberati *et al.*, 2000): phononic propagation in a fluid is described by a wave equation that, under appropriate conditions, can be interpreted as propagation in an effective relativistic curved spacetime background, the spacetime metric being entirely determined by the physical properties of the fluid under study, namely, its density and flow velocity. Unruh urged a specific motivation (Unruh, 1981) for examining the hydrodynamic analogue of an event horizon (Misner *et al.*, 1993), namely, that as an experimentally and theoretically accessible phenomenon it might shed some light on the Hawking effect (Hawking, 1974, 1975) (thermal radiation from black holes, stationary insofar as the back reaction is negligible). In particular, one would like to gain insight into the role in the Hawking process of ultrahigh frequencies (Corley, 1998; Corley and Jacobson, 1999; Jacobson, 1991; Unruh, 1995).

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An event horizon for sound waves appears in principle wherever there is a surface through which a fluid flows at the speed of sound, the flow being subsonic on one side of the surface and supersonic on the other. There is a close analogy between sound propagation on a background hydrodynamic flow, and field propagation in a curved spacetime; and although hydrodynamics is only a long-wavelength effective theory for physical (super)fluids, so also field theory in curved spacetime is to be considered a long-wavelength approximation to quantum gravity (Unruh, 1995; Visser, 1998a,b). Determining whether and how sonic black holes radiate sound, in a full calculation beyond the hydrodynamic approximation or in an actual experiment, can thus offer some suggestions about black hole radiance and its sensitivity to high frequency physics (beyond the Planck scale). The possibility that such high frequencies might have consequences for observably low frequency phenomena is one of the main reasons that black holes have deserved much attention: there is reason to expect that an event horizon can act as a microscope, giving us a view into physics on scales below the Planck length. This is because modes coming from an event horizon are redshifted into the low-energy regime as they propagate out to be observed far away from the black hole. Conversely, if we imagine tracking the observed signal back towards its source, the closer we get to the horizon, the shorter the wave length of the signal must be, until at the very horizon we must either reach infinite energy scales, or encounter a breakdown in general relativity and quantum field theory in curved spacetime (for a review, see e.g., Jacobson, 1999).

In understanding this problem, hydrodynamic and condensed matter analogs of black holes may offer some of the experimental guidance otherwise difficult to obtain in the case of gravity (for a review, see, e.g., Jacobson, 1999). Under appropriate conditions and approximations (which can be basically summarized in the requirement the wave lengths of the perturbations be sufficiently large), the propagation of collective fluctuations (phonons) admits an effective general relativistic description, in terms of a spacetime metric. This long-wavelength regime would correspond analogically to quantum field theory in curved spacetime. The effective phonon metric may describe black holes, as in general relativity, and so a phonon Hawking effect may be possible; and certainly the problem of arbitrarily high frequencies at the horizon is also present. But in this case, when at short wave-lengths the metric approximation is no longer valid and a more microscopic theory must be used instead, the accurate microscopic theory is actually known.

Dilute-gas Bose–Einstein condensates, which can now be easily manipulated and controlled, both experimentally (Anderson *et al.*, 1995; Davis *et al.*, 1995), and theoretically (Dalfovo *et al.*, 1999), provide a hydrodynamic system whose microscopic theory is actually tractable enough that we can make reliable calculations from first principles. As we will argue, trapped bosons at ultralow temperature can indeed provide an analogue to a black-hole spacetime. Similar analogues have been proposed in other contexts, such as superfluid helium (Jacobson and Volovik, 1998; Ruutu *et al.*, 1996; Volovik, 1999a,b), solid state physics (Reznik, 1997), and optics (Leonhardt and Piwnicki, 1999, 2000); but the outstanding recent experimental progress in cooling, manipulating, and controlling atoms (Burger *et al.*, 1999; Deng *et al.*, 1999; Matthews *et al.*, 1999) make Bose–Einstein condensates an especially powerful tool for this kind of investigation.

Here, we discuss the theoretical framework and propose an experiment to create the analog of a black hole in the laboratory and simulate its radiative instabilities (Garay *et al.*, 2000, 2001). The basic challenge of our proposal is to keep the trapped Bose–Einstein gas sufficiently cold and well isolated to maintain a locally supersonic flow long enough to observe its intrinsic dynamics. Detecting thermal phonons radiating from the horizons would obviously be a difficult additional problem, since such radiation would be indistinguishable from many other possible heating effects. This further difficulty does not arise in our proposal, however, because the black-hole radiation we predict is not quasistationary, but grows exponentially under appropriate conditions. It should therefore be observable in the next generation of atom traps, and may also raise new issues in the theory of gravitational black holes.

2. SONIC BLACK HOLES IN CONDENSATES

A Bose–Einstein condensate is the ground state of a second quantized many body Hamiltonian for N interacting bosons trapped by an external potential $V_{\text{ext}}(\mathbf{x})$ (Dalfovo *et al.*, 1999). At zero temperature, when the number of atoms is large and the atomic interactions are sufficiently small, almost all the atoms are in the same single-particle quantum state (mean field) $\Psi(\mathbf{x}, t)$, even if the system is slightly perturbed. The evolution of Ψ is then given by the well-known Gross–Pitaevskii equation, which can be written as

$$i\hbar\partial_t\Psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}} + \frac{4\pi a\hbar^2}{m}|\Psi|^2\right)\Psi,$$

where m is the mass of the individual atoms and a is the scattering length. The wave function of the condensate is normalized to the total number of atoms $\int d^3\mathbf{x}|\Psi(\mathbf{x}, t)|^2 = N$.

Our concern is the propagation of small collective perturbations of the condensate, around a background stationary state

$$\Psi_s(\mathbf{x}, t) = \sqrt{\rho(\mathbf{x})}\, e^{i\vartheta(\mathbf{x})}\, e^{-i\mu t/\hbar},$$

where μ is the chemical potential. Thus it is only necessary that it be possible, in any external potential $V_{\text{ext}}(\mathbf{x})$ that can be generated, to create a condensate in this state. Indeed, many realistic techniques for "quantum state engineering," to create designer potentials and bring condensates into specific states, have been proposed, and even implemented successfully (Burger *et al.*, 1999; Deng *et al.*, 1999; Matthews *et al.*, 1999); and our simulations indicate that currently known techniques should suffice to generate the condensate states that we propose.

Perturbations about the stationary state $\Psi_s(\mathbf{x}, t)$ obey the Bogoliubov system of two coupled differential equations. Within the regime of validity of the hydrodynamic (Thomas–Fermi) approximation (Dalfovo *et al.*, 1999), low frequency perturbations are essentially just waves of (zero) sound. Indeed, the Bogoliubov equations may be reduced to a single second-order equation for the condensate phase perturbation ϕ, which can be written in terms of the perturbations of the wave function $\psi \equiv \Psi - \Psi_s$ as $\phi = -i\psi/\Psi_s$. This differential equation has the form of a relativistic wave equation $\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 0$, with $g = \det g_{\mu\nu}$, in an effective curved spacetime with the metric $g_{\mu\nu}$ being entirely determined by the local speed of sound $c(\mathbf{x}) \equiv \frac{\hbar}{m}\sqrt{4\pi a\rho(\mathbf{x})}$, and the background stationary velocity field $\mathbf{v} \equiv \frac{\hbar}{m}\nabla\vartheta$. Up to a conformal factor, this effective metric has the form

$$(g_{\mu\nu}) = \begin{pmatrix} -(c^2 - \mathbf{v}^2) & -\mathbf{v}^{\mathrm{T}} \\ -\mathbf{v} & \mathbf{1} \end{pmatrix}.$$

This class of metrics can possess event horizons. For instance, if an effective sink for atoms is generated at the center of a spherical trap (such as by an atom laser out-coupling technique; Andrews *et al.*, 1997; Bloch *et al.*, 1999; Hagley *et al.*, 1999), and if the radial potential profile is suitably arranged, we can produce densities $\rho(r)$ and flow velocities $\mathbf{v}(\mathbf{x}) = -v(r)\mathbf{x}/r$ such that the quantity $c^2 - \mathbf{v}^2$ vanishes at a radius $r = r_h$, being negative inside and positive outside. The sphere at radius r_h is a sonic event horizon completely analogous to those appearing in general relativistic black holes, in the sense that sonic perturbations cannot propagate through this surface in the outward direction (Unruh, 1981, 1995; Visser, 1998a,b). This can be seen explicitly by writing the equation for the radial null geodesics of the metric $g_{\mu\nu}$: $\dot{r}_\pm = -v \pm c$. The ingoing null geodesic $r_-(t)$ is not affected by the presence of the horizon and can cross it in a finite coordinate time t. The outgoing null geodesic $r_+(t)$ on the other hand needs an infinite amount of time to leave the horizon since $\dot{r}_+ = 0$ at the horizon. The physical mechanism of the sonic black hole is quite simple: inside the horizon, the background flow speed v is larger than the local speed of sound c, and so sound waves are inexorably dragged inwards.

In fact there are two conditions that must hold for this dragged sound picture to be accurate. Wavelengths larger than the black hole itself will of course not be dragged in, but merely diffracted around it. And perturbations must have wavelengths

$$\lambda \gg \frac{\pi\hbar}{mc}, \quad \frac{\pi\hbar}{mc\sqrt{|1 - v/c|}}.$$

Otherwise they do not behave as sound waves since they lie outside the regime of validity of the hydrodynamic approximation. These short-wavelength modes must be described by the full Bogoliubov equations, which allow signals to propagate faster than the local sound speed, and thus permit escape from sonic black holes. So, to identify a condensate state Ψ_s as a sonic black hole, there must exist modes with wavelengths larger than these lower limits, but also smaller than the black hole size.

As it stands, this description is incomplete. The condensate flows continually inwards and therefore at $r = 0$ there must be a sink that takes atoms out of the condensate. Otherwise, the continuity equation $\nabla(\rho\mathbf{v}) = 0$, which must hold for stationary configurations, will be violated. From the physical point of view, such a sink can be accomplished by means of an outcoupler laser beam at the origin. Such outcouplers are the basic mechanisms for making trapped condensates into "atom lasers," and they have already been demonstrated experimentally by several groups. A tightly focused laser pulse changes the internal state of the atoms at a particular point in the trap, and can also be made to give them a large momentum impulse. This ejects them so rapidly through the always dilute condensate cloud that they do not significantly disturb it; effectively, they simply disappear.

We have analyzed several specific systems that may be suitable theoretical models for future experiments, and have found that the qualitative behavior is analogous in all of them. Black holes that require atom sinks are both theoretically and experimentally more involved, however; moreover, maintaining a steady supersonic flow into a sink may require either a very large condensate or some means of replenishment. We will therefore first discuss an alternative configuration that may be experimentally more accessible and whose description is particularly simple: a condensate in a very thin ring that effectively behaves as a periodic one-dimensional system (Fig. 1(a)). Under conditions that we will discuss, the supersonic region in a ring may be bounded by two horizons: a black hole horizon through which phonons cannot exit, and a "white hole" horizon through which they cannot enter. Then we will analyze another simple one-dimensional model of a long straight condensate with an atom sink at the center (Fig. 1(b)).

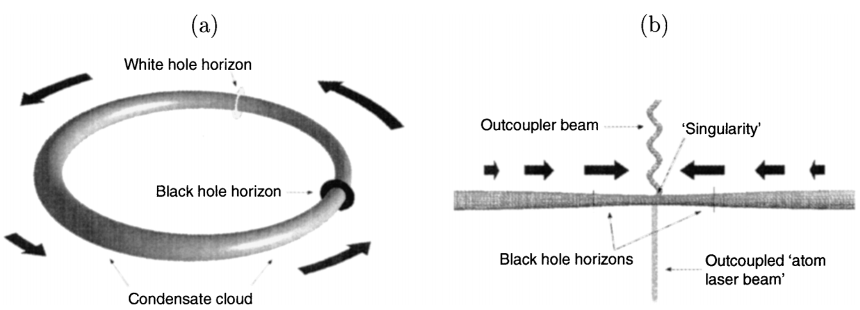

Fig. 1. (a) The tight ring-shaped configuration, with both black and white horizons and no singularity. (b) The tight cigar-shaped configuration, with two black hole horizons and a "singularity" where condensate is outcoupled. Arrows indicate condensate flow velocity, with longer arrows for faster flow.

3. BLACK/WHITE HOLES IN A RING

In a sufficiently tight ring-shaped external potential of radius R, motion in radial (r) and axial (z) cylindrical coordinates is effectively frozen. We can then write the wave function as $\Psi(z, r, \theta, \tau) = f(z, r)\Phi(\theta, \tau)$ and normalize Φ to the number of atoms in the condensate $\int_0^{2\pi} d\theta |\Phi(\theta)|^2 = N$, where, with the azimuthal coordinate θ, we have introduced the dimensionless time $\tau = (\hbar/mR^2)t$. The Gross–Pitaevskii equation thus becomes effectively one-dimensional:

$$i\partial_\tau \Phi = \left(-\frac{1}{2}\partial_\theta^2 + \mathcal{V}_{\rm ext} + \frac{\mathcal{U}}{N}|\Phi|^2\right)\Phi,$$

where $\mathcal{U} \equiv 4\pi a N R^2 \int dz\, dr\, r|f(z, r)|^4$ and $\mathcal{V}_{\rm ext}(\theta)$ is the dimensionless effective potential (in which we have already included the chemical potential) that results from the dimensional reduction. The stationary solution can then be written as $\Phi_s(\theta, \tau) = \sqrt{\rho(\theta)}\, e^{i\int d\theta\, v(\theta)}$ and the local dimensionless angular speed of sound as $c(\theta) = \sqrt{\mathcal{U}\rho(\theta)/N}$. Periodic boundary conditions around the ring require the "winding number" $w \equiv (1/2\pi)\int_0^{2\pi} d\theta\, v(\theta)$ to be an integer.

The qualitative behavior of horizons in this system is well represented by the three-parameter family of speed-of-sound and flow-velocity fields (both of them related by the continuity equation $\partial_\theta(\rho v) = 0$)

$$c(\theta)^2 = \frac{\mathcal{U}}{2\pi}(1 + b\,\cos\theta), \quad v(\theta) = \frac{\mathcal{U}w\sqrt{1-b^2}}{2\pi c(\theta)^2},$$

where $b \in [0, 1]$. Requiring that $\Phi_s(\theta, \tau)$ be a stationary solution to Gross–Pitaevskii equation then determines how the trapping potential must be modulated as a function of θ. All the properties of the condensate, including whether and where it has sonic horizons, and whether or not they are stable, are thus functions of $\mathcal{U}$, b, and w. For instance, if we require that the horizons be located at $\theta_h = \pm\pi/2$, which imposes the relation $\mathcal{U} = 2\pi w^2(1 - b^2)$, then we must have $c^2 - v^2$ positive for $\theta \in (-\pi/2, \pi/2)$, zero at $\theta_h = \pm\pi/2$, and negative otherwise, provided that $\mathcal{U} < 2\pi w^2$. The further requirement that perturbations on wavelengths shorter than the inner and the outer regions are indeed phononic implies $\mathcal{U} \gg 2\pi$, which in turn requires $w \gg 1$ and $1 \gg b \gg 1/w^2$. In fact, detailed analysis shows that $w \gtrsim 5$ is sufficient.

The mere existence of a black-hole solution does not necessarily mean that it is physically realizable: it should also be stable over sufficiently long time scales. Since stability must be checked for perturbations on all wavelengths, the full Bogoliubov (Dalfovo *et al.*, 1999) spectrum must be determined. For large black holes within large, slowly varying condensates, this Bogoliubov problem may be solved using WKB methods that closely resemble those used for solving relativistic field theories in true black-hole spacetimes (Corley, 1998; Corley and Jacobson, 1999). A detailed adaptation of these methods to the Bogoliubov problem will be presented elsewhere (Garay *et al.*, unpublished). The results are qualitatively similar to those we have found for black holes in finite traps with low winding number, where we have resorted to numerical methods because, in these cases, WKB techniques may fail for just those modes that threaten to be unstable.

Our numerical approach for our three-parameter family of black/white holes in the ring-shaped condensate has been to write the Bogoliubov equations in discrete Fourier space, and then truncate the resulting infinite-dimensional eigenvalue problem. Indeed, writing the wave funtion as $\Phi = \Phi_s + \varphi\, e^{i\int d\theta\, v(\theta)}$, decomposing the perturbation φ in discrete modes

$$\varphi(\theta, \tau) = \sum_{\omega,n} e^{-i\omega\tau}\, e^{in\theta} A_{\omega,n} u_{\omega,n}(\theta) + e^{i\omega^*\tau}\, e^{-in\theta} A^*_{\omega,n} v^*_{\omega,n}(\theta),$$

and substituting into the Gross–Pitaevskii equation we obtain a set of algebraic equations for the modes $u_{\omega,n}$ and $v_{\omega,n}$.

Eliminating Fourier components above a sufficiently high cutoff Q has negligible effect on possible instabilities, which can be shown to occur at relatively long wavelengths. We face then an eigenvalue problem for a $2(Q+1)\times 2(Q+1)$ matrix whose numerical solution, together with the normalization condition $\int d\theta (u^*_{\omega^*,n} u_{\omega',n'} - v^*_{\omega^*,n} v_{\omega',n'}) = \delta_{nn'}\delta_{\omega\omega'}$, provides the allowed frequencies. Real negative eigenfrequencies for modes of positive norm are always present, which means that black-hole configurations are energetically unstable, as expected. This feature is inherent in supersonic flow, since the speed of sound is also the Landau critical velocity. In a sufficiently cold and dilute condensate, however, the time scale for dissipation may in principle be made very long, and so these energetic instabilities need not be problematic (Fedichev and Shlyapnikov, 1999).

More serious are dynamical instabilities, which occur for modes with complex eigenfrequencies. Since the Bogoliubov theory is based on a quantized Hamiltonian that is Hermitian, there are certainly no complex energy eigenvalues; but the natural frequencies of normal modes can indeed be complex (in which case the usual rule, that energy eigenvalues are $\hbar(n + 1/2)$ times the mode frequencies, simply breaks down). A discussion of the quantum mechanics of dynamical instability is presented in Section 5; for the time being, it suffices to note that complex eigenfrequencies are indeed genuine physical phenomena, and by no means a numerical artifact. For sufficiently high values of the cutoff (e.g., $Q \geq 25$ in our calculations), the complex eigenfrequencies obtained from the truncated eigenvalue problem become independent of the cutoff within the numerical error. The existence and rapidity of dynamical instabilities depend sensitively on $(\mathcal{U}, b, w)$. For instance, see Fig. 2 for a contour plot of the maximum of the absolute values of the imaginary part of all eigenfrequencies for $w = 7$, showing that the regions of instability are long, thin fingers in the $(\mathcal{U}, b)$ plane. It also shows the size of the largest absolute value of the instabilities for each point on the dashed curve. This figure illustrates the important fact that the size of the imaginary parts, which

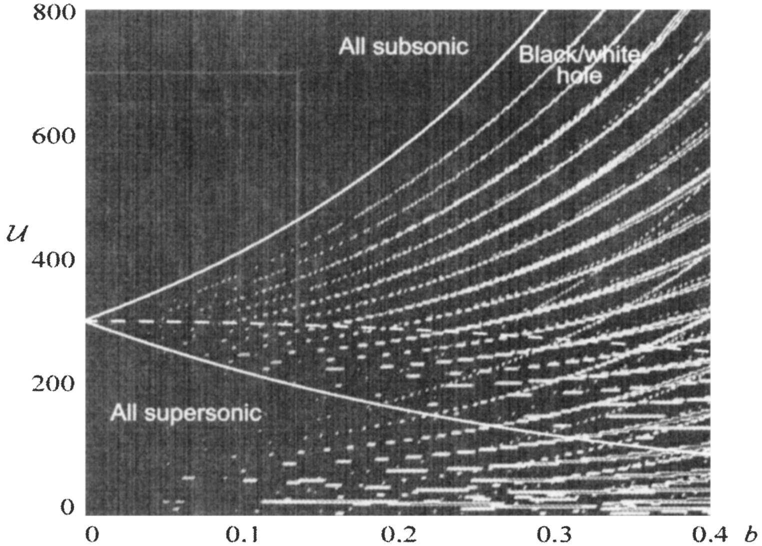

Fig. 2. Stability diagram for winding number $w = 7$. Solid dark-grey areas represent the regions of stability. Smaller plots at higher resolution confirm that the unstable "fingers" are actually smooth and unbroken. Points on the dashed curve are states with horizons at $\theta_h = \pm\pi/2$, so that the black/white hole fills half the ring.

gives the rate of the instabilities, increases starting from zero, quite rapidly with b, although they remain small as compared with the real parts.

The stability diagram of Fig. 2 suggests a strategy for creating a sonic black hole from an initial stable state. Within the upper subsonic region, the vertical axis $b = 0$ corresponds to a homogeneous persistent current in a ring, which can in principle be created using different techniques (Dum *et al.*, 1998; Williams and Holland, 1999). Gradually changing $\mathcal{U}$ and b, it is possible to move from such an initial state to a black/white hole state, along a path lying almost entirely within the stable region, and only passing briefly through instabilities where they are sufficiently small to cause no difficulty.

Indeed, we have simulated this process of adiabatic creation of a sonic black/white hole by solving numerically (using the split operator method) the time-dependent Gross–Pitaevskii equation that provides the evolution of the condensate when the parameters of the trapping potential change so as to move the condensate state along various paths in parameter space. One of these paths is shown in Fig. 2 (light-grey solid line): we start with a current at $w = 7$, $b = 0$, and sufficiently high $\mathcal{U}$ (Fig. 3(a)); we then increase b adiabatically keeping $\mathcal{U}$ fixed until an appropriate value is reached (Fig. 3(b)); finally, keeping b constant, we decrease $\mathcal{U}$ adiabatically (which can be physically implemented by decreasing

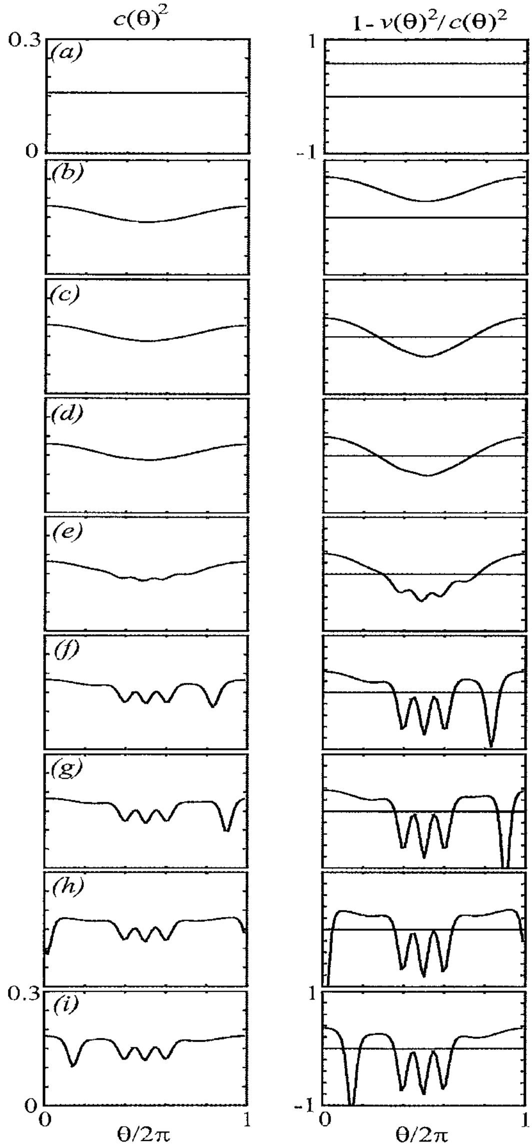

Fig. 3. Simulation of creation of a stable black/white hole and subsequent evolution into an unstable region. Figure (a)–(d) are snapshots taken at the initial time (a), at an intermediate time, still within the subsonic region (b), when the black/white holes of maximum size is approached (c), and after a long time in that configuration (d). Then the parameters are changed along the dashed curve of Fig.2 to enter an unstable region (e) and kept there (f)–(i). It can be observed that a perturbation grows at the black hole horizon and travels rightwards until it enters the white hole horizon.

the radius of the ring trap), until we meet the dashed contour for black holes of comfortable size (Fig.3(c)). Our simulations confirm that the small instabilities that briefly appear in the process of creation do not disrupt the adiabatic evolution. The final quantum state of the condensate, obtained by this procedure, indeed represents a stable black/white hole. We have further checked the stability of this final configuration by numerically solving the Gross–Pitaevskii equation for very long periods of time (as compared with any characteristic time scale of the condensate) and for fixed values of the trap parameters. This evolution reflects the fact that no imaginary frequencies are present, as predicted from the mode analysis, and that the final state is indeed stationary (Fig.3(d)). Once the black/white hole has been created, one could further change the parameters $(\mathcal{U}, b)$ so as to move between the unstable "fingers" into a stable region of higher b (a deeper hole).

Instead of navigating the stable region of parameter space, one could deliberately enter an unstable region (Fig. 3(e)–(i)). In the this case, the black hole should disappear in an explosion of phonons, which may be easy to detect experimentally. Such an event might be related to the evaporation process suggested for real black holes, in the sense that pairs of quasiparticles are created near the horizon in both positive and negative energy modes. We will explain this point briefly in Section 5.

4. SINK-GENERATED BLACK HOLES

We present a simple model that exhibits the main qualitative features of more general situations and that can be studied analytically. Although in this model, we study a condensate of infinite size, in more realistic models or experiments, it will suffice to take condensates that are sufficiently large, since the stability pattern is not significantly affected by the (large but finite) size of the condensate.

Let us consider a tight cigar-shaped condensate of infinite size such that the motion in the (y, z) plane is effectively frozen. In appropriate dimensionless units, the effectively one-dimensional Gross–Pitaevskii equation thus becomes

$$i\partial_\tau \Phi = \left(-\frac{1}{2}\partial_x^2 + \mathcal{V}_{\rm ext} + \mathcal{U}|\Phi|^2\right)\Phi.$$

In this equation, $\mathcal{V}_{\rm ext}$ is the dimensionless effective potential that results from the dimensional reduction, which already includes the chemical potential.

In order to obtain a black-hole configuration, let us choose the potential $\mathcal{V}_{\rm ext}$ so that it produces a profile for the speed of sound $c(x) = \sqrt{\mathcal{U}\rho(x)}$ of the form (Fig. 4)

$$c(x) = \begin{cases} c_0, & |x| < L \\ c_0[1 + (\sigma - 1)x/\epsilon], & L < |x| < L + \epsilon \\ \sigma c_0, & L + \epsilon < |x| \end{cases},$$

with $\sigma > 1$, and a flow velocity in the inward direction. The continuity equation

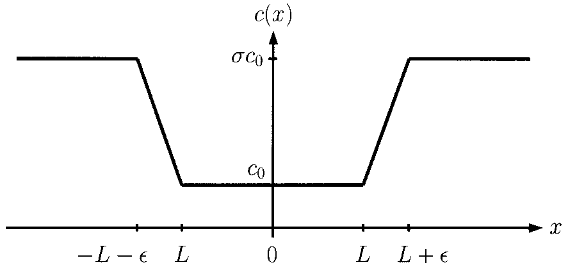

Fig. 4. Profile for the speed of sound $c(x)$ for the one-dimensional sink-generated black hole.

then provides the flow velocity profile

$$v(x) = -\frac{v_0 c_0^2}{c(x)^2}\frac{x}{|x|},$$

where v_0 is the absolute value of the flow velocity in the inner region.

As it stand this model fails to fulfill the continuity equation at $x = 0$. In order to take this into account, we will also introduce a sink of atoms at $x = 0$ that takes atoms out of the condensate (this can be physically implemented by means of a laser). From the mathematical point of view, it can be modeled by an additional term in the equation of the form $-iE\delta(x)$ that indeed induces loss of atoms at $x = 0$. Equivalently, it can be represented by boundary conditions of the form

$$\begin{aligned}\Phi(0^+, \tau) - \Phi(0^-, \tau) &= 0,\\ \Phi'(0^+, \tau) - \Phi'(0^-, \tau) &= -2iE\Phi(0, \tau),\end{aligned} \tag{1}$$

which determine the flow velocity inside in terms of the characteristics of the outcoupler laser, namely, $v_0 = E$. As a further simplifying assumption, we will assume that $v_0\epsilon \ll 1$.

Perturbations ψ around this stationary state $\Phi_s = \sqrt{\rho}\, e^{i\int v(x)\,dx}$, such that $\Phi = \Phi_s + \psi$ (note that for convenience we have chosen a different convention as compared with the ring in which $\Phi = \Phi_s + \varphi\, e^{i\int v}$) must satisfy the boundary conditions (1) and the equation

$$i\dot{\psi} = -\frac{1}{2}\psi'' + (c^2 - v^2/2 + c''/2c)\psi + c^2\, e^{2i\int^x v}\psi^*.$$

Let us now expand the perturbation ψ in modes

$$\psi = \sum_{\omega,k}\left[A_{\omega,k}u_{\omega,k}(x)\, e^{-i\omega\tau} + A^*_{\omega,k}v_{\omega,k}(x)^*\, e^{i\omega^*\tau}\right].$$

Then, the modes $u_{\omega,k}(x)$ and $v_{\omega,k}(x)$ satisfy, in each region, the Bogoliubov equations

$$\omega u_{\omega,k} = -\frac{1}{2}u''_{\omega,k} + (c^2 - v^2/2)u_{\omega,k} + c^2\, e^{2i\int^x v} v_{\omega,k},$$
$$\omega v_{\omega,k} = \frac{1}{2}v''_{\omega,k} - (c^2 - v^2/2)v_{\omega,k} - c^2\, e^{-2i\int^x v} u_{\omega,k}. \tag{2}$$

The intermediate regions $L < |x| < L + \epsilon$ provide the connection between the perturbation modes in the inner and outer regions. Once these connection formulas have been established, in the limit of small ϵ, we will only need to study the inside and outside modes and their relation through such formulas. The case of an abrupt horizon, in which the background condensate velocity is steeply and linearly ramped within a very short interval, is obviously quite special; and it does not particularly resemble the horizon of a large black hole in Einsteinian gravity. But the connection formula that we derive for this case will qualitatively resemble those that are obtained, with considerably more technical effort, for smoother horizons (Garay *et al.*, unpublished). And the results we will obtain for the global Bogoliubov spectrum of the condensate black hole will indeed be representative of more generic cases.

The singular character of c''/c at $|x| = L, L + \epsilon$ can be substituted by matching conditions at $|x| = L,\ L + \epsilon$. Furthermore, the symmetry of the problem allows us to study only the region $x > 0$. These matching conditions for the perturbation ψ, together with the form of the modes in the region $L < x < L + \epsilon$ provide the connection formulas between the inside ($|x| < L$) and outside ($|x| > L$) modes (from now on we will drop the subindex ω):

$$u_{\mathrm{in},k}(L) = -\epsilon u'_{\mathrm{out},k}(L) + \frac{1}{\sigma}u_{\mathrm{out},k}(L),$$
$$u'_{\mathrm{in},k}(L) = \sigma u'_{\mathrm{out},k}(L),$$

and likewise for the modes $v_{\mathrm{in,out}}$.

In each of the regions (inside and outside), we can write

$$u_k(x) = u_k\, e^{i(k-|v|)(x-L)}, \quad v_k(x) = v_k\, e^{i(k+|v|)(x-L)}.$$

Upon substitution of this expansion into the Bogoliubov equations (2), we obtain, for each region, the following set of algebraic equations:

$$h_k^- u_k + c^2 v_k = 0, \quad c^2 u_k + h_k^+ v_k = 0,$$

where $h_k^{\pm} = k^2/2 + c^2 \pm (k|v| + \omega)$. For these equations to have a solution, the determinant must vanish thus providing the dispersion relation

$$k^4/4 + (c^2 - v^2)k^2 - 2\omega|v|k - \omega^2 = 0, \tag{3}$$

which, for fixed ω, is a fourth-order equation for k. For each of the four solutions u_k and v_k must be related by

$$v_k = h_k u_k \quad \text{with} \quad h_k = -\frac{1}{c^2}(k^2/2 + c^2 - k|v| - \omega).$$

The constant coefficients u_k can be regarded as normalization constants and will be set to unity. The possible solutions to the dispersion relation depend on whether ω is real or complex. For complex frequencies, all four solutions are pure complex, two of them with positive imaginary part and two of them with negative one. For real frequencies, on the other hand, outside ($c^2 > v^2$), there are two real and two complex conjugate solutions. Of these two complex solutions, only one is allowed (the one with $\mathrm{Im}(k) > 0$) because the other grows exponentially. Inside ($c^2 < v^2$), for $\omega > \omega_{\max}$ there are two real and two complex conjugate solutions; for $\omega < \omega_{\max}$ there are four real solutions; the value $\omega = \omega_{\max}$ is a bifurcating point.

Since we are interested in the existence of dynamical instabilities, we will concentrate in the case in which ω is complex. Then, as we have seen, the dispersion equation (3) has four complex solutions for k in each region. Inside, all four solutions $k_{\mathrm{in},i}$, $i = 1 \cdots 4$ are in principle possible but outside those with $\mathrm{Im}(k_{\mathrm{out}}) < 0$ will increase exponentially. Therefore, up to corrections coming from the finite size of the condensate, which we ignore here, only modes associated with $k_{\mathrm{out},\alpha}$, $\alpha = 1, 2$ such that $\mathrm{Im}(k_{\mathrm{out},\alpha}) > 0$ are allowed. Each mode $u_{\mathrm{out},\alpha}(x) = e^{i(k_{\mathrm{out},\alpha} - v_0/\sigma^2)(x-L)}$ will match a linear combination $u_{\mathrm{in},\alpha}(x) = \sum_i F_{\alpha i} u_{\mathrm{in},i}(x)$ of modes $u_{\mathrm{in},i}(x)$ inside, and similarly for $v_{\mathrm{out},\alpha}$ and $v_{\mathrm{in},\alpha}$.

After some straightforward calculations, it can be seen that these connecting coefficients $F_{\alpha i}$ are given by $F_{\alpha i} = \sum_j (M^{-1})_{ij} C_{\alpha j}$, where

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ k^-_{\mathrm{in},1} & k^-_{\mathrm{in},2} & k^-_{\mathrm{in},3} & k^-_{\mathrm{in},4} \\ h_{\mathrm{in},1} & h_{\mathrm{in},2} & h_{\mathrm{in},3} & h_{\mathrm{in},4} \\ h_{\mathrm{in},1}k^+_{\mathrm{in},1} & h_{\mathrm{in},2}k^+_{\mathrm{in},2} & h_{\mathrm{in},3}k^+_{\mathrm{in},3} & h_{\mathrm{in},4}k^+_{\mathrm{in},4} \end{pmatrix},$$

$$C_\alpha = \begin{pmatrix} 1/\sigma - i\epsilon k^-_{\mathrm{out},\alpha} \\ \sigma k^-_{\mathrm{out},\alpha} \\ (1/\sigma - i\epsilon k^+_{\mathrm{out},\alpha}) h_{\mathrm{out},\alpha} \\ \sigma k^+_{\mathrm{out},\alpha} h_{\mathrm{out},\alpha} \end{pmatrix}.$$

In these equations, $k^\pm_{\mathrm{in},i} = k_{\mathrm{in},i} \pm v_0$ and $k^\pm_{\mathrm{out},\alpha} = k_{\mathrm{out},\alpha} \pm v_0/\sigma^2$.

We have already found the modes in the inner and outer region as well as their relation. To determine which (complex-frequency) modes will be present, there only remains to impose the boundary conditions dictated by the presence of the sink at $x = 0$.

As we have already mentioned, the symmetry of the system under reflection ($x \to -x$) allows us to study only the region $x > 0$, provided that we study the even and odd perturbations separately. For odd fluctuations [$\psi_o(x, \tau) = -\psi_o(-x, \tau)$], the boundary conditions (1) become $\psi_o(0, \tau) = 0$ at all times τ. This implies that the u and v components of ψ must separately satisfy the boundary condition. Since we can have any linear combination of the two solutions that decay outside the horizon, we therefore have a two-by-two matrix constraint. The condition that a nonzero solution exists is that

$$\det \begin{pmatrix} u_{\text{in},1}(0) & u_{\text{in},2}(0) \\ v_{\text{in},1}(0) & v_{\text{in},2}(0) \end{pmatrix} = 0,$$

and therefore

$$\sum_{ij} F_{1i} F_{2j} (h_{\text{in},i} - h_{\text{in},j})\, e^{-i(k_{\text{in},i}+k_{\text{in},j})L} = 0\ . \tag{4}$$

For even fluctuations [$\psi_e(x, \tau) = \psi_e(-x, \tau)$], the boundary conditions (1) become $\psi_e'(0, \tau) + i v_0 \psi_e(0, \tau) = 0$, which implies that

$$\sum_{ij} F_{1i} F_{2j} (h_{\text{in},i} - h_{\text{in},j}) k_{\text{in},i} k_{\text{in},j}\, e^{-i(k_{\text{in},i}+k_{\text{in},j})L} = 0. \tag{5}$$

For fixed $L, \mathcal{U}, v_0$, and σ, the quantities F, h_{in}, and k_{in} that appear in Eqs. (4) and (5) are only functions of ω. Therefore, the solutions to these equations are all the possible complex eigenfrequencies, which depend on the free parameters that determine the model, namely, the size $2L$ of the inner region, the speed of sound inside c_0, the relative change of the speed of sound between the inner and the outer regions σ, and the flow velocity inside v_0 (related to the characteristics of the outcoupler laser). In practice, there are also other parameters of the condensate such as its size $2D$ (which has been made arbitrarily large) and the size of the intermediate regions ϵ (which has been made arbitrarily small).

Equations (4) and (5) can be solved numerically for different values of the parameters $\sigma, \mathcal{U}, v_0$, and L. The numerical method employed is the following. The equations above have the form $f(\omega; \sigma, \mathcal{U}, v_0, L) = 0$, where f and ω are both in general complex. We plot contours of constant absolute value of f in the complex ω plane; where $|f|$ approaches zero, we have an eigenfrequency.

The distribution of complex solutions in the complex ω plane depends on the size of the inner region L, for given $\sigma, \mathcal{U}$, and v_0. Direct inspection of the numerical results shows that the number of instabilities increases by one when the black-hole size L is increased by π/k_0 where $k_0 = \sqrt{v_0^2 - c_0^2}$. More explicitly, for L smaller than $\pi/k_0 - \delta$ (δ being much smaller than π/k_0) there are no complex eigenfrequencies; for $(L + \delta)k_0/\pi \in [n, n+1]$ with $n = 1, 2 \ldots$, we have n complex solutions except for $L = (n + 1/2)\pi/k_0$, where we find $n - 1$ complex solutions instead of n (i.e., there is one mode for which $\text{Im}(\omega) = 0$ within numerical resolution). This can be easily interpreted qualitatively since the unstable modes are basically the bound states in the black hole, and the highest wave number k on the positive norm upper branch, for the barely bound state with $\omega \to 0^-$, is exactly k_0. So the threshold is simply when the well becomes big enough to have a bound state; the small δ displacement comes in because the horizon is not exactly a hard wall; and similarly for the extra bound state every π/k_0. Thus, stability can only be achieved for small sizes of the inner region, $L \lesssim \pi/k_0$. As we discussed in Section 2, the wave length $2\pi/k$ of the perturbations must be smaller than this size, which implies $k > 2\pi/L \gtrsim 2k_0$. However, for these perturbations the hydrodynamic approximation, which requires $k \lesssim 2k_0$, is not valid. Therefore, there are no stable black-hole configurations in a strict sense. The sizes of the imaginary parts of the complex solutions decrease as the size L of the interior of the black hole increases. Thus, although a larger hole has more unstable modes, it is actually less unstable (and might even became quasistable in the sense that its instability time scale would be longer than the experimental duration).

5. QUASIPARTICLE PAIR CREATION

In the language of second quantization, the perturbation field operator ψ satisfies the linear equation

$$i\hbar\dot{\psi} = h_0(\mathbf{x})\psi + mc(\mathbf{x})^2\, e^{2i\vartheta(\mathbf{x})}\psi^\dagger,$$

where $h_0(\mathbf{x}) = -\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}} + 2mc(\mathbf{x})^2 - \mu$. Taking into account that $[\psi(\mathbf{x}), \psi^\dagger(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}')$, this equation can be written as $i\hbar\dot{\psi} = [\psi, H]$, where the Bogoliubov Hamiltonian is

$$H = \int d\mathbf{x} \left[-\psi^\dagger h_0(\mathbf{x})\psi + \frac{1}{2}mc(\mathbf{x})^2\, (e^{2i\vartheta(\mathbf{x})}\psi^\dagger\psi^\dagger + e^{-2i\vartheta(\mathbf{x})}\psi\psi) \right].$$

The Hermiticity of the Bogoliubov linearized Hamiltonian implies that eigenmodes with complex frequencies appear always in dual pairs, whose frequencies are complex conjugate. Indeed, expanding the perturbation field operator ψ in modes

$$\psi(\mathbf{x}, t) = \sum_k \left[e^{-i\hbar\omega_k t} A_{\omega_k,k} u_{\omega_k,k}(\mathbf{x}) + e^{i\hbar\omega_k^* t} A^*_{\omega_k,k} v^*_{\omega_k,k}(\mathbf{x}) \right],$$

together with the normalization condition

$$\int d\mathbf{x} \big(u^*_{\omega_k^*,k} u_{\omega_{k'},k'} - v^*_{\omega_k^*,k} v_{\omega_{k'},k'} \big) = \delta_{kk'},$$

we can write the linearized Hamiltonian as

$$H = \hbar \sum_k (\omega_k A^\dagger_{\omega_k^*,k} A_{\omega_k,k} + \omega_k^* A^\dagger_{\omega_k,k} A_{\omega_k^*,k}),$$

the only nonvanishing commutators among these operators being $[A_{\omega_k,k}, A^\dagger_{\omega_{k'}^*,k'}] = \delta_{kk'}$. The asterisk on the subscript is important: the mode with frequency ω_k^* is a different mode from the one with frequency ω_k, and $A^\dagger_{\omega^*_k,k}$ is not the Hermitian conjugate of $A_{\omega_k,k}$. It is therefore clear that none of these operators is actually a harmonic oscillator creation or annihilation operator in the usual sense. However, the linear combinations

$$a_k = \frac{1}{\sqrt{2}}(A_{\omega_k,k} + A_{\omega^*_k,k}), \quad b_k = \frac{i}{\sqrt{2}}(A^\dagger_{\omega_k,k} + A^\dagger_{\omega^*_k,k})$$

and their Hermitian conjugates are true annhilation and creation operators, with the standard commutation relations, and in terms of these the Bogoliubov Hamiltonian becomes

$$H = \hbar \sum_k \left[\mathrm{Re}(\omega_k)(a^\dagger_k a_k - b^\dagger_k b_k) - \mathrm{Im}(\omega_k)(a^\dagger_k b^\dagger_k + a_k b_k)\right].$$

This interaction obviously leads to self-amplifying creation of positive and negative frequency pairs, which resembles the usual process of Hawking evaporation.

That the Hamiltonian H is unbounded from below does not indicate anything unphysical about our model (Fulling, 1989; Kang, 1996; Schroer and Swieca, 1970): we have simply linearized about an unstable excited state of the nonlinear full Hamiltonian, which is bounded from below. Real negative frequencies ω indicate energetic instabilities, whereby the system will decay in the presence of dissipation. Complex frequencies, on the other hand, indicate dynamical instabilities. Classically, a dynamically unstable system will exponentially diverge from the initial stationary state if is perturbed, even without dissipation. Quantum mechanically, a dynamically unstable system has no normalizable stationary states, and it can be easily checked that this is indeed the case for the Bogoliubov Hamiltonian H above. If an initially stable system is driven into a state that is stationary but dynamically unstable at the classical (mean field) level, the initial state will have had finite Hilbert space norm, and hence under unitary evolution the final state will have the same norm. Thus it will not be a stationary state; one may say that quantum fluctuations will always trigger the dynamical instability. For a logarithmically long period of time, however, the linearized theory will still remain valid and, in this sense, our linearized description of quantum dynamical instabilities is sound.

6. CONCLUSIONS

We have seen that dilute Bose–Einstein condensates admit, under appropriate conditions, configurations that closely ressemble gravitational black-holes. We have analyzed in detail the case of a condensate in a ring trap, and proposed a realistic scheme for adiabatically creating stable sonic black/white holes and we have seen that there exist stable and unstable black-hole configurations. We have also studied a model for a sink-generated sonic black hole in an infinite one-dimensional condensate. The dynamical instabilities can be interpreted as coming from quasiparticle pair creation, as in the well-known suggested mechanism for black-hole evaporation. Generalizations to spherical or quasi-two-dimensional traps, with flows generated by laser-driven atom sinks, should also be possible, and should behave similarly. While our analysis has been limited to Bogoliubov theory, the further theoretical problems of back reaction and other corrections to simple mean field theory should be more tractable for condensates than for other systems analogous to black holes. And we expect that experiments along the lines we have proposed, including both creation and evaporation of sonic black holes, can be performed with state-of-the-art or planned technology.

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